JOURNAL OF MATHEMATICAL ANALYSIS AND APPLICATIONS **81**, 189–203 (1981)

Characteristic Wave Fronts in Magnetohydrodynamics

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Submitted by C. L. Dolph

The influence of magnetic field on the process of steepening or flattening of the characteristic wave fronts in a plane and cylindrically symmetric motion of an ideal plasma is investigated. This aspect of the problem has not been considered until now. Remarkable differences between plane, cylindrical diverging, and cylindrical converging waves are discovered. For instance, when the adiabatic index γ is 2, the magnetic field does not affect the behaviour of plane waves, but does affect cylindrical waves. As the field strength increases, the time t_c taken for the shock formation varies monotonically for plane waves, while for cylindrical waves, in some situations t_c exhibits a unique minimum for diverging waves and a unique maximum for converging waves. For cylindrical converging waves, a shock formation takes place if and only if, γ and the field strength are restricted to certain finite intervals. Moreover, t_c is bounded in all cases except for cylindrical diverging waves. The discontinuity in the velocity gradient at the wave front is shown to satisfy a Bernoulli-type equation. The discussion of the solutions of such equations reported in the literature is shown to be incomplete, and three general theorems are established.

1. INTRODUCTION

A number of problems relating to wave propagation in magnetohydrodynamics and in collisionless plasma have been studied previously, in particular by Ludford [1], Bohachevsky [2], Jeffrey and Taniuti [3], Kato *et al.* [4], and Gopalkrishna [5] among others. However, the effect of magnetic field strength on the steepening or flattening tendencies of the wave fronts has not been investigated until now.

The purpose of this paper is to study the effects of magnetic field strength, the initial value of the discontinuity associated with the wave front, the adiabatic index γ, and the initial wave-front curvature (in case of cylindrical waves) on the growth and decay properties of the characteristic wave fronts in plane and in cylindrically symmetric motions of a uniform plasma. The plasma is assumed to be an ideal gas with infinite electrical conductivity, and to be permeated by a magnetic field orthogonal to the trajectories of gas particles.

0022-247X/81/050189–15$02.00/0

Let γ denote the adiabatic index, s_0 the magnitude of the initial discontinuity, k_0 the initial wave-front curvature, and ε_0 a quantity depending upon the magnetic field strength such that $\varepsilon_0 = 1$ when the magnetic field is absent and ε_0 increases to ∞ as the magnetic field strength increases to ∞. Moreover, let t_c be the shock formation time. The formation of a shock wave and the time t_c for the shock wave to form depend upon γ, s_0, k_0, and ε_0 in an interesting fashion. Since the detailed analysis performed in Section 3 is somewhat complicated, the following summary of the results will be found useful to compare the behaviour of different types of waves. In addition to the range $1 < \gamma \leqslant 2$, which has been of more physical interest until now, we have also considered the cases $\gamma \leqslant 1$ and $\gamma > 2$ for the sake of completeness of the analysis.

(a) *Plane waves.* (i) All expansive waves ultimately decay to zero, while compressive waves grow without bound in a finite time. (ii) When $\gamma < 2$, both the decaying of an expansive wave and the steepening of a compressive wave are enhanced by the presence of a magnetic field; further, an increase in magnetic field causes t_c to decrease, and t_c approaches a finite non-zero limiting value as $\varepsilon_0 \to \infty$ (iii) When $\gamma = 2$, the magnetic field has no effect on the growth and decay properties of the wave front. (iv) When $\gamma > 2$, both the decaying of an expansive wave and the steepening of a compressive wave are slowed down by the presence of a magnetic field; further, an increase in magnetic field causes t_c to increase, and t_c approaches a finite non-zero limiting value as $\varepsilon_0 \to \infty$.

(b) *Cylindrical diverging waves.* (i) All expansive waves attenuate, while compressive waves grow without bound. (ii) The decay rate of an expansive wave is enhanced by an increase in the magnetic field irrespective of the value of γ. (iii) An increase in k_0 causes an increase in t_c. (iv) When $\gamma < 2$, there are two situations depending on the range of γ, $|s_0|$, and k_0 such that in one case, t_c is a monotonic increasing function of ε_0, and in the other case, t_c has a unique minimum at a value (say, ε^*) of ε_0 such that $\varepsilon^* > 1$ and t_c decreases over the interval $(1, \varepsilon^*)$ and increases as ε_0 increases from ε^*, and $t_c \to \infty$ as $\varepsilon_0 \to \infty$. Moreover, when γ is fixed, an increase in the ratio $k/|s_0|$ causes ε^* to increase, and when the ratio $k_0/|s_0|$ remains constant, an increase in γ leads to a decrease in ε^*. (v) When $\gamma \geqslant 2$, an increase in the magnetic field strength causes t_c to increase, and $t_c \to \infty$ as $\varepsilon_0 \to \infty$.

(c) *Cylindrical converging waves.* (i) All expansive waves form a focus, while not all compressive waves lead to a shock. In fact, there exists a positive critical value s_c of the initial discontinuity such that if $|s_0| < s_c$, a compressive wave forms a focus but not the shock; if $|s_0| = s_c$, a compressive wave forms a shock and focus simultaneously; and if $|s_0| > s_c$, then a compressive wave terminates into a shock before the formation of the focus. Further, a compressive wave terminates into a shock if γ and ε_0 lie in certain intervals; for values of γ and ε_0 beyond this range, a focus is formed but no shock formation takes place; and for the remaining values of γ and ε_0, both a shock and a focus are formed simultaneously. (ii) An increase in k_0 causes t_c to decrease. (iii) When $\gamma \leqslant 2$, an increase in ε_0 (which is restricted to a suitable interval for a shock to be formed) causes t_c to decrease. (iv) When $\gamma > 2$, there are two situations depending on the range of γ and ε_0 and the restrictions on $|s_0|$ and k_0 such that in one case t_c is a decreasing function of ε_0, while in the other case t_c has a unique maximum at some ε^* belonging to the range of ε_0 with $\varepsilon^* > 1$. When γ is fixed, an increase in the ratio $k_0/|s_0|$ causes ε^* to decrease, but if the ratio $k_0/|s_0|$ remains constant, an increase in γ leads to an increase in ε^*.

The discontinuity at the wave front is shown to satisfy a Bernoulli-type equation which occurs frequently in studies of acceleration waves and other phenomena. The question naturally arises as to whether the general results known for such an equation are applicable to our problem. We find that some of the known theorems, reported in Chen [6, Chap. 3] need certain modifications: the last section of the present paper contains three of these modified theorems. These theorems are adequate to characterise the situations when the wave ultimately damps out, forms a shock, attains a stable wave form, or forms a focus.

2. Behaviour at the Wave Front

Equations which describe the one-dimensional planar ($\nu = 0$) or cylindrically symmetrical ($\nu = 1$) motion of a perfect plasma in the presence of a transverse magnetic field can be written down in the familiar form

$$\rho_t + u\rho_x + \rho u_x + \nu x^{-1} \rho u = 0, \tag{1}$$

$$\rho u_t + \rho u u_x + p_x + \mu H H_x = 0, \tag{2}$$

$$H_t + u H_x + H u_x + \nu x^{-1} u H = 0, \tag{3}$$

$$p_t + u p_x + \gamma p(u_x + \nu x^{-1} u) = 0, \tag{4}$$

where u is the gas velocity, p the pressure, ρ the density, H the magnetic field strength, γ the adiabatic index, μ the magnetic permeability, t the time, and x is the single spatial coordinate being either axial in flows with planar geometry, or radial in cylindrically symmetric flows. Letter subscripts denote partial differentiation unless stated otherwise. The system of equations (1) to (4) possesses four families of characteristics [7], two of which,

$dx/dt = u \pm c$, represent waves propagating in the $\pm x$ direction with the magnetoacoustic speed c given by

$$c = (a^2 + b^2)^{1/2}; \qquad a^2 = \gamma p/\rho, \qquad b^2 = \mu H^2/\rho,$$

and the remaining two form a set of double characteristic $dx/dt = u$, representing the particle path or trajectory.

In terms of the characteristics (a wave tag ϕ and a particle tag ψ) as the reference coordinate system, the system of equations (1) to (4) can be transformed into the following equivalent system:

$$(c\rho_\phi - \rho u_\phi)\, t_\psi + \rho u_\psi t_\phi + \nu x^{-1} \rho u c t_\phi t_\psi = 0, \tag{5}$$

$$(p_\phi + \mu H H_\phi - \rho c u_\phi) t_\psi - (p_\psi + \mu H H_\psi)\, t_\phi = 0, \tag{6}$$

$$p_\psi + \mu H H_\psi + c \rho u_\psi = -\nu x^{-1} u c^2 \rho t_\psi, \tag{7}$$

$$(c H_\phi - H u_\phi)\, t_\psi + H u_\psi t_\phi + \nu x^{-1} c u H t_\phi t_\psi = 0, \tag{8}$$

$$x_\phi = u t_\phi, \tag{9}$$

$$x_\psi = (u + c)\, t_\psi. \tag{10}$$

Any dependent variable, f, say, will transform so that

$$f_x = (f_\phi t_\psi - f_\psi t_\phi)/J, \tag{11}$$

$$f_t + u f_x = -c f_\phi t_\psi / J, \tag{12}$$

where J is the Jacobian of the transformation namely $x_\phi t_\psi - x_\psi t_\phi$. It follows from Eqs. (9) and (10) that

$$J = -c t_\phi t_\psi. \tag{13}$$

Since "doubling up" or overlapping of fluid particles is prohibited from physical considerations, $t_\psi \neq 0$. Consequently $J = 0$, if, and only if, $t_\phi = 0$ when two adjoining characteristics merge into a shock wave.

If one now considers the case in which the wave head $x = x(t)$, which is defined as the location of a discontinuity in u_x, is an outgoing characteristic, then the boundary conditions to the system in terms of ϕ and ψ are

$$p = p_0,\ \rho = \rho_0,\ H = H_0,\ u = 0,\ t = \psi \qquad \text{at} \quad \phi = 0, \tag{14}$$

$$p_\psi = \rho_\psi = H_\psi = u_\psi = 0,\ t_\psi = 1 \qquad \text{at} \quad \phi = 0, \tag{15}$$

where the subscript 0 denotes constant values in the undisturbed region ahead of the wave front.

Equations (5) to (10) yield

$$\rho_\phi = \left(\frac{\rho_0}{c_0}\right) u_\phi, p_\phi = \left(\frac{\rho_0 a_0^2}{c_0}\right) u_\phi, H_\phi = \left(\frac{H_0}{c_0}\right) u_\phi,$$

$$x_\phi = 0, x_\psi = c_0 \qquad \text{at} \quad \phi = 0. \tag{16}$$

Now, to compute u_x at the wave front $\phi = 0$, we set $s = u_x|_{\phi=0}$ and invoke (11) and (13) which yield

$$s = -u_\phi/(c_0 t_\phi). \tag{17}$$

Differentiating (7) and (10) with respect to ϕ, and (6) and (9) with respect to ψ and using the foregoing results, we find that at the wave front $\phi = 0$

$$u_{\phi\psi} = -(\nu c_0/2x)\, u_\phi, \tag{18}$$

$$t_{\phi\psi} = -(\Lambda_0/c_0)\, u_\phi, \tag{19}$$

where

$$\Lambda_0 = 1.5 + \{(\gamma-2)/(2\varepsilon_0^2)\}; \qquad \varepsilon_0^2 = (c_0/a_0)^2 = 1 + (b_0/a_0)^2. \tag{20}$$

Integrating (18) with respect to ψ on the line of constant $\phi(=0)$, we obtain

$$u_\phi = u_{\phi_0}(x/x_0)^{-\nu/2}, \tag{21}$$

where u_{φ_0} and x_0 are respectively the values of u_ϕ and x at $t = 0$. Making use of (21) in (19) and integrating with respect to ψ, we get

$$t_\phi = t_{\phi_0} - \frac{x_0 \Lambda_0 u_{\phi_0}}{c_0^2(1-\nu/2)} \{(x/x_0)^{1-\nu/2} - 1\}, \tag{22}$$

where t_{ϕ_0} is the values of t_ϕ at $t = 0$.

Using (21) and (22) in (17), we finally obtain

$$s = \frac{s_0(x/x_0)^{-\nu/2}}{1 + c_0^{-1}(1-\nu/2)^{-1}\Lambda_0 x_0 s_0\{(x/x_0)^{1-\nu/2} - 1\}}, \tag{23}$$

where $s_0(\neq 0)$ is the value of s at $t = 0$. It is evident from (23) that if $s_0 > 0$ (i.e., an expansive wave front), s will decrease monotonically in time. But if $s_0 < 0$ (i.e., a compressive wave front), the denominator in (23) may vanish at some finite time; $|s|$ must tend to ∞ at such a time, and this signifies the appearance of a shock wave. The coincidence of this behaviour with the vanishing of t_ϕ, and hence of the Jacobian of transformation is clear from Eqs. (13), (17), and (22).

3. Discussion

In each of the following three situations, we find essentially two possibilities: expansive waves ultimately damp out, while compressive waves may grow into a shock wave in a finite time t_c. A detailed study of shock formation time t_c and its dependence on the four parameters (the adiabatic index γ, the initial wave front curvature $k_0 = 1/|x_0|$, the initial discontinuity s_0, and the magnetic field intensity ε_0) reveals interesting differences in the three situations.

(I) *Plane Waves* ($\nu = 0$)

For plane waves, Eq. (23) reduces to

$$s = s_0(1 + \Lambda_0 s_0 t)^{-1}, \tag{24}$$

where Λ_0 is as in (20). Therefore, if $s_0 > 0$ then the wave decays ($s \to 0$ as $t \to \infty$), and if $s_0 < 0$ then a shock is formed ($|s| \to \infty$ as $t \to t_c$) at a finite time

$$t_c = (\Lambda_0 |s_0|)^{-1}. \tag{25}$$

Consider the effect of γ and ε_0 on s. Equation (24) shows that when $\gamma < 2$, then both the decaying of an expansive wave and the steepening of a compressive wave are enhanced by the presence of a magnetic field ($\varepsilon_0^2 > 1$). This feature is somewhat different from other effects such as geometrical convergence (or divergence) and dissipation which increase one rate but diminish the other. Here the steepening or flattening of the wave front is the result of a non-linear pulse-shaping mechanism and not of energy loss, and this mechanism is stronger when a magnetic field is added.

When $\gamma = 2$, the magnetic field drops out of Eq. (24), and thus it has no effect on the steepening or flattening of the wave front. It may be noted that in the Chew–Goldberger–Low model of a plasma, $\gamma = 2$ for waves transverse to the magnetic field (involving $p_\perp$). When $\gamma > 2$, both the decaying of an expansive wave and the steepening of a compressive wave are slowed down by the presence of a magnetic field.

As the magnetic field increases, Eq. (25) shows that the time t_c taken for the shock formation decreases or increases, respectively, according as $\gamma < 2$, or $\gamma > 2$; in both cases $t_c \to 2/(3|s_0|)$ as $\varepsilon_0 \to \infty$. Thus, even a very strong magnetic field cannot offset the tendency of a plane compressive wave to grow into a shock. We shall see presently that this is not true for cylindrical waves.

(II) *Cylindrical Diverging Waves* ($\nu = 1, x_0 > 0$)

Equation (23) may be written in dimensionless form as

$$s/s_0 = (1 + \varepsilon_0 |\Phi| \tau)^{-1/2} \{1 + 2\Phi^{-1}\varepsilon_0^{-1} \Lambda_0((1 + \varepsilon_0 |\Phi| \tau)^{1/2} - 1)\}^{-1}, \quad (26)$$

where

$$\tau = t|s_0|, \qquad \Phi = a_0/(x_0 s_0). \quad (27)$$

In the present case, $|\Phi| = a_0/(x_0|s_0|) = a_0 k_0/|s_0|$.

From (26), if $s_0 > 0$ then the wave decays ($s \to 0$ as $t \to \infty$), and if $s_0 < 0$ then the wave steepens up into a shock wave ($|s| \to \infty$ as $t \to t_c$) at a finite $t_c = \tau_c/|s_0|$, where

$$\tau_c = \Lambda_0^{-1}\{1 + (|\Phi|\varepsilon_0/4\Lambda_0)\}. \quad (28)$$

First consider the effect of γ and ε_0 on the behaviour of t_c or, equivalently, on τ_c. We establish that

(i) $\tau_c \to \infty$ as $\varepsilon_0 \to \infty$; in fact, $\lim_{\varepsilon_0 \to \infty}(\tau_c/\varepsilon_0) = |\Phi|/9$.

(ii) Let $\gamma < 2$. If either $\gamma > 1.4$ and $|\Phi| \geqslant 4(2-\gamma)(1+\gamma)/(5\gamma - 7)$, or $\gamma \geqslant [4 - 5|\Phi| + \{(4 - 5|\Phi|)^2 + 16(7|\Phi| + 8)\}^{1/2}]/8$, then τ_c increases as ε_0 increases.

(iii) Let $\gamma < 2$. If $\gamma \leqslant 1.4$, or, $\gamma > 1.4$ and $|\Phi| < 4(2-\gamma)(1+\gamma)/(5\gamma - 7)$, or $\gamma < [4 - 5|\Phi| + \{(4 - 5|\Phi|)^2 + 16(7|\Phi| + 8)\}^{1/2}]/8$, then there is an $\varepsilon^* > 1$ such that τ_c decreases as ε_0 increases from 1 to ε^*, and τ_c increases as ε_0 increases from ε^* to ∞, so that τ_c has a *unique minimum* at ε^*, where ε^* is the unique positive solution of the equation

$$(2-\gamma)(|\Phi|\,\varepsilon_0^3)^{-1} + \{3\varepsilon_0^2(2-\gamma)^{-1} - 1\}^{-1} = \tfrac{1}{4}. \quad (29)$$

In fact, $\varepsilon^* > \{5(2-\gamma)/3\}^{1/2}$.

(iv) Consider the ε^* of (iii) above where τ_c is a minimum. If one of the quantities γ or $|\Phi|$ is fixed and the other increases, then ε^* decreases.

(v) If $\gamma \geqslant 2$, then τ_c increases as ε_0 increases.

Statement (i) follows from (20). Statements (ii), (iii), and (v) are proved by considering the sign of $\partial\tau_c/\partial\varepsilon_0$ as follows. A simple computation yields

$$\Lambda_0^2 \left(\frac{\partial \tau_c}{\partial \varepsilon_0}\right) = \frac{|\Phi|}{4} + (\gamma - 2)\left(\frac{1}{\varepsilon_0^3} + \frac{|\Phi|}{3\varepsilon_0^2 + \gamma - 2}\right), \quad (30)$$

which is positive for all $\varepsilon_0 \geqslant 1$ if $\gamma \geqslant 2$, so that (v) follows. If $\gamma < 2$, the right side of (30) increases with ε_0 and is ultimately positive. Hence, if $\partial\tau_c/\partial\varepsilon_0$ is non-negative at $\varepsilon_0 = 1$, then $\partial\tau_c/\partial\varepsilon_0$ is positive for all $\varepsilon_0 > 1$, while if $\partial\tau_c/\partial\varepsilon_0$

is negative at $\varepsilon_0 = 1$, then there must exist an $\varepsilon^* > 1$ such that $\partial\tau_c/\partial\varepsilon_0$ is negative for $1 < \varepsilon_0 < \varepsilon^*$ and positive for $\varepsilon_0 > \varepsilon^*$, so that τ_c has a minimum at ε^*. The right side of (30) vanishes at ε^*, which leads to (29). To examine the sign of $\partial\tau_c/\partial\varepsilon_0$ at $\varepsilon_0 = 1$, we find from (30) that

$$\left(\Lambda_0^2 \frac{\partial\tau_c}{\partial\varepsilon_0}\right)_{\varepsilon_0=1} = \frac{|\Phi|}{4}(5\gamma - 7) - (2-\gamma)(1+\gamma)$$
$$= \gamma^2 - \left(1 - \frac{5|\Phi|}{4}\right)\gamma - \left(\frac{7|\Phi|}{4} + 2\right). \tag{31}$$

The middle expression in (31) is negative if either $5\gamma \leqslant 7$, or if $\gamma > 1.4$ and $|\Phi| < 4(2-\gamma)(1+\gamma)/(5\gamma - 7)$. The last expression in (31) is negative if, and only if, γ lies between the two roots of the quadratic; however, one of the roots is negative and the other is positive, so that γ should be less than the positive root. This completes the proof of (ii) and (iii). The last statement of (iii) follows by solving (29) for $|\Phi|$. To prove (iv), note that the left side of (29) decreases if any one of the three quantities $|\Phi|$, γ, and ε_0 increases, keeping the other two quantities fixed. Thus, for instance, if γ is fixed and $|\Phi|$ increases, then ε^* must decrease in order to keep the function constant.

The effect on t_c of variation in the initial wave-front curvature k_0 is as follows: t_c increases when the initial curvature of the wave front increases, because $s_0(\partial t_c/\partial|\Phi|) = \partial\tau_c/\partial|\Phi|$ is positive from (28). Similarly, an increase in $|s_0|$ leads to a decrease in t_c. Note that if $s_0 < 0$, then the wave always leads to a shock no matter how small $|s_0|$ is. Moreover, the shock formation time t_c for a cylindrical diverging wave is greater than that for a plane wave, with the same $|s_0|$ and ε_0, in view of (28).

(III) *Cylindrical Converging Waves* ($\nu = 1, x_0 < 0$).

Here (23) may be written as

$$s = \frac{s_0(1 - k_0 c_0 t)^{-1/2}}{1 + (s_0/s_c)\{1 - (1 - k_0 c_0 t)^{1/2}\}}, \tag{32}$$

where $k_0 = 1/|x_0|$ is the initial wave-front curvature, and

$$s_c = c_0 k_0/(2\Lambda_0). \tag{33}$$

The numerator on the right side of (32) becomes unbounded at t^*, where $t^* = 1/(k_0 c_0)$. The denominator on the right side of (32) varies monotonically from 1 to $1 + (s_0/s_c)$ as t increases from 0 to t^*. Thus, if $s_0 > 0$, then $s \to \infty$ as $t \to t^*$, leading to the formation of a focus. However, when $s_0 < 0$, there are three possibilities; if $|s_0| < s_c$ then $|s| \to \infty$ as $t \to t^*$, which corresponds to the formation of a focus but not a shock; if $|s_0| = s_c$, it follows from (32) that $|s| \to \infty$ as $t \to t^*$, which corresponds to the simultaneous formation of a shock and a focus; and, finally, if $|s_0| > s_c$, the denominator in (32) vanishes at some finite time $t_c < t^*$, i.e., $|s| \to \infty$ as $t \to t_c$, where

$$t_c = (c_0 k_0)^{-1} \left\{ 1 - \left(1 - \frac{s_c}{|s_0|} \right)^2 \right\} = \frac{s_c}{k_0 c_0 |s_0|} \left(2 - \frac{s_c}{|s_0|} \right), \tag{34}$$

so that in this case, a shock is formed before the focus ($t_c < t^*$).

We see that, unlike the previous situations, a shock does not necessarily form if $s_0 < 0$. Let us first examine the necessary restrictions on γ, k_0, and ε_0 for a shock to occur. In view of (20) and (33), the condition $s_c \leqslant |s_0|$ is equivalent to the condition

$$3\varepsilon_0^{-1} + (\gamma - 2)\, \varepsilon_0^{-3} \geqslant |\Phi|, \tag{35}$$

where Φ is as in (27); in the present case, $|\Phi| = a_0/(|x_0||s_0|) = a_0 k_0/|s_0|$. The equality in (35) represents the simultaneous formation of a shock and a focus.

Suppose that $\gamma \geqslant 2$. The left side of (35) decreases monotonically from $1 + \gamma$ to zero as ε_0 increases from 1 to ∞. Therefore, shock formation cannot take place if $1 + \gamma < |\Phi|$, while shock occurs for all ε_0 in a certain range $1 \leqslant \varepsilon_0 < \varepsilon'$ if $1 + \gamma \geqslant |\Phi|$, where ε' is that value of ε_0 which satisfies the equality in (35), and shock does not occur when $\varepsilon_0 > \varepsilon'$.

Suppose now that $\gamma < 2$. The derivative of the left side of (35) is $3(2 - \gamma - \varepsilon_0^2)/\varepsilon_0^4$ which is positive if $\varepsilon_0 < (2-\gamma)^{1/2}$ and negative if $\varepsilon_0 > (2-\gamma)^{1/2}$, so that the left side of (35) increases with ε_0 over $0 < \varepsilon_0 < (2-\gamma)^{1/2}$ and decreases over $(2-\gamma)^{1/2} < \varepsilon_0 < \infty$, with the maximum value $2(2-\gamma)^{-1/2}$ occurring when $\varepsilon_0 = (2-\gamma)^{1/2}$. Since ε_0 is restricted to the range $\varepsilon_0 \geqslant 1$, the inequality (35) can now be tackled easily. If $(2-\gamma)^{1/2} \leqslant 1$, i.e., $\gamma \geqslant 1$, then the left side of (35) decreases over $\varepsilon_0 \geqslant 1$ and we have the same situation as for $\gamma \geqslant 2$ above. If $(2-\gamma)^{1/2} > 1$, i.e., $\gamma < 1$, then shock does not occur if $2(2-\gamma)^{-1/2} < |\Phi|$, while shock occurs when, and only when, ε_0 lies in a certain range $\varepsilon_1 \leqslant \varepsilon_0 \leqslant \varepsilon'$ if $2(2-\gamma)^{-1/2} \geqslant |\Phi|$. In the last possibility the equality in (35) has two positive roots, the larger of the roots is denoted by ε', and ε_1 is the smaller of the roots if $\varepsilon_1 > 1$, otherwise we take $\varepsilon_1 = 1$; thus $\varepsilon_1 = 1$ if, and only if, $1 + \gamma \geqslant |\Phi|$, which follows by evaluating (35) at $\varepsilon_0 = 1$.

We may summarize the restrictions necessary for shock formation as follows. In view of (33), (27), and (20), a cylindrical converging wave grows into a shock if, and only if, $\Lambda_0 \geqslant |\Phi| \varepsilon_0/2$; if the inequality holds then the shock occurs before the focus does (i.e., $t_c < t^*$), and if the equality holds then the shock occurs simultaneously with the focus (i.e., $t_c = t^*$).

Equivalently, we have the following restrictions on γ and the magnetic field ε_0: (i) Let $\gamma \geqslant 1$. If $(1+\gamma) < |\Phi|$, there is no shock formation. If $(1+\gamma) \geqslant |\Phi|$, then there is a unique $\varepsilon' > 1$ satisfying the equality in (35) such that there is no shock formation if $\varepsilon_0 > \varepsilon'$, a shock is formed before the focus if $1 \leqslant \varepsilon_0 < \varepsilon'$ and a shock is formed together with the focus if $\varepsilon_0 = \varepsilon'$. (ii) Let $\gamma < 1$. If $2(2-\gamma)^{-1/2} < |\Phi|$, there is no shock formation for all ε_0. If $2(2-\gamma)^{-1/2} = |\Phi|$, there is a unique $\varepsilon' \geqslant 1$ satisfying the equality in (35), such that when $\varepsilon_0 = \varepsilon'$ there is simultaneous formation of shock and focus, while for all $\varepsilon_0 \neq \varepsilon'$, there is no shock formation. If $2(2-\gamma)^{-12} > |\Phi|$ the equality in (35) has precisely two positive roots, one of which (denoted by ε') exceeds unity, and we define the number ε_1 to be the smaller of the roots, or 1 according as, respectively $(1+\gamma)$ is less than or not less than $|\Phi|$; then there is shock formation if, and only if, ε_0 lies in the interval $\varepsilon_1 \leqslant \varepsilon_0 \leqslant \varepsilon'$. Moreover, the shock is formed before the focus if $\varepsilon_1 < \varepsilon_0 < \varepsilon'$ while both the shock and focus are formed together when either $\varepsilon_0 = \varepsilon'$ and $(1+\gamma) \leqslant |\Phi|$, or, $\varepsilon_0 = \varepsilon'$.

We are now in a position to discuss the effect of s_0, k_0, and ε_0 on the shock formation time t_c. As we did for diverging waves in (II) above, we vary one of the parameters, keeping the other fixed. It is clear from the above discussion that k_0^{-1} or $|s_0|$ can be increased indefinitely and the shock will still be formed, while ε_0 is to be restricted to a suitable finite interval $(1, \varepsilon')$ or $(\varepsilon_1, \varepsilon')$.

The shock formation time t_c decreases as the curvature k_0 increases, which becomes clear upon substituting (33) into (34); t_c decreases as the initial discontinuity $|s_0|$ increases, because the last expression in (34) involves the product $y(2-y)$ which is an increasing function of y in $0 \leqslant y \leqslant 1$, with $y = s_c/|s_0|$.

To examine the effect of increasing ε_0, we consider the sign of $\partial t_c/\partial \varepsilon_0$. A simple computation yields

$$\varepsilon_0^3 |s_0| \Lambda_0^2 \left(\frac{\partial t_c}{\partial \varepsilon_0}\right) = -\left(1 - \frac{s_c}{|s_0|}\right)(2-\gamma) - \frac{s_c \Lambda_0 \varepsilon_0^2}{2|s_0|}. \tag{36}$$

If $\gamma \leqslant 2$, then t_c decreases as ε_0 increases, because $\partial t_c/\partial \varepsilon_0$ is negative. If $\gamma > 2$, we know from (i) above that $(1+\gamma) \geqslant |\Phi|$ and that ε_0 is restricted to the interval $[1, \varepsilon']$, where ε' is the unique solution to the equality in (35), so that $s_c/|s_0|$ increases to 1 as ε_0 increases to ε'. Therefore, the right side of (36) is negative when $\varepsilon_0 = \varepsilon'$, because the first term vanishes at $\varepsilon_0 = \varepsilon'$. Moreover, the right side of (36) is a decreasing function of ε_0 over $1 \leqslant \varepsilon_0 \leqslant \varepsilon'$. At $\varepsilon_0 = 1$, in view of (27), the right side of (36) becomes

$$-\{(|\Phi|/4)(5\gamma - 7) - (\gamma - 2)(1+\gamma)\}/(1+\gamma). \tag{37}$$

We have already studied the sign of a similar quantity in the argument which followed (31). A repetition of the same argument establishes the following conclusions:

(i) If $\gamma \leqslant 2$, then t_c decreases as ε_0 increases.

(ii) Suppose that $\gamma > 2$. If $|\Phi| \geqslant 4(\gamma-2)(1+\gamma)/(5\gamma-7)$, or equivalently, γ does not lie between the two quantities $[4+5|\Phi| \pm \{(4+5|\Phi|)^2 - 16(7|\Phi|-8)\}^{1/2}]/8$, then t_c decreases as ε_0 increases (if a shock is formed, i.e., t_c exists, then $|\Phi| \leqslant 1+\gamma$).

(iii) Suppose that $\gamma > 2$. If $|\Phi| < 4(\gamma-2)(1+\gamma)/(5\gamma-7)$, or, equivalently, γ lies in the open interval with end points at $[4+5|\Phi| \pm \{(4+5|\Phi|)^2 - 16(7|\Phi|-8)\}^{1/2}]/8$, then there is an $\varepsilon^* > 1$ such that t_c increases as ε_0 increases from 1 to ε^*, and t_c decreases as ε_0 increases from ε^* to ε'. Thus t_c has a maximum at ε^*, and minima at $\varepsilon_0 = 1$ and at $\varepsilon_0 = \varepsilon'$; also ε^* satisfies the equation

$$(3 + (\gamma-2)/\varepsilon_0^2)(|\Phi|\varepsilon_0)^{-1} - 3\varepsilon_0^2(4(\gamma-2))^{-1} = \tfrac{5}{4},$$

which expresses the fact that (37) vanishes at $\varepsilon_0 = \varepsilon^*$.

(iv) In (iii) above, consider the ε^* where t_c is a maximum. If $|\Phi|$ increases, and γ is fixed, then the corresponding ε^* decreases. If γ increases and $|\Phi|$ is fixed, then the corresponding ε^* increases.

4. Behaviour of the Solution of Bernoulli Equation

Differentiating (17) with respect to ψ and using (18), (19), and the condition $t = \psi$ at $\phi = 0$, we obtain the following Bernoulli-type equation for the discontinuity s with $\mu = \nu c_0/(2x)$ and $\beta = -\Lambda_0$,

$$\frac{ds}{dt} = -\mu s + \beta s^2. \tag{38}$$

The most general analysis known to date of the solution of (38), with μ, β as functions of t, and β of constant sign, is given in [8–10]. All these results, without any modification are quoted as Theorems 3.2.2 to 3.2.6 in a book by Chen [6], and the results of [8] are also quoted in the *Handbuch der Physik* [11]. Unfortunately some of these theorems are incomplete and one of them is incorrect. Several workers (e.g., [12–14]) after deriving such an equation, do refer to these only known results, while many workers (e.g., [5, 15–18]) do not seem to be aware of these results and therefore analyse their problems only for simple cases. We feel that such a situation is a handicap to workers. The present section modifies Theorems 3.2.2, 3.2.4, and 3.2.6 stated in [6].

As in [6, 8–10], let the function $\beta(t)$ be of constant sign, i.e., sgn $\beta(t) = +1$ for all t, or $\operatorname{sgn} \beta(t) = -1$ for all t.

The modified results are as follows:

THEOREM 1. *Consider Eq.* (38) *with* $s(0) \neq 0$. *Suppose that* $\mu(t)$ *and* $\beta(t)$ *are integrable on every finite sub-interval of* $[0, \infty)$, *the function* $\beta(t)$ *is of constant sign on* $[0, \infty)$ *and* $\operatorname{sgn} s(0) = -\operatorname{sgn} \beta(t)$. *Let* $\liminf_{t\to\infty} |\beta(t)| \neq 0$ *and define* $\lambda(t) = \mu(t)/\beta(t)$.

(a) *If* $\operatorname{sgn} \lambda(t) = -\operatorname{sgn}\ s(0)$ *for all sufficiently large* t, *then* $\lim_{t\to\infty} s(t) = 0$;

(b) *If* λ *is bounded above* (*resp., below*) *or tends to a non-negative* (*resp., non-positive*) *finite or infinite limit* L, *the same is true for any solution* $s(t) > 0$ (*resp.,* $s(t) < 0$).

Proof. Statement (b) is Theorem 3.2.2 of Chen; in this case, $\operatorname{sgn} \lambda(t) = \operatorname{sgn} s(0)$ for all sufficiently large t. It is therefore sufficient to prove (a). The solution of (38) is

$$s(t) = \{\exp(-F(t))\} \Big/ \left\{1/s(0) - \int_0^t \beta(\tau) \exp(-F(\tau))\, d\tau\right\}, \tag{39}$$

where

$$F(t) = \int_0^t \mu(\tau)\, d\tau. \tag{40}$$

Since $\operatorname{sgn} \beta(t) = -\operatorname{sgn} s(0)$, we may write (39) as

$$s(t) = \{\operatorname{sgn} s(0)\} \{\exp(-F(t))\} \Big/ \left\{1/|s(0)| + \int_0^t |\beta(\tau)| \exp(-F(\tau))\, d\tau\right\}. \tag{41}$$

Now, $\operatorname{sgn} \lambda(t) = -\operatorname{sgn} s(0) = \operatorname{sgn} \beta(t)$ implies that $\operatorname{sgn}\ \mu(t) = +1$ for all sufficiently large t, say for $t \geqslant T$, so that $\exp(-F(t))$ monotonically decreases as t increases when $t \geqslant T$. The integral in the denominator of (41) is an increasing function of t, therefore, it either converges or diverges to $+\infty$; if it converges then $\exp(-F(t)) \to 0$ and hence $s(t) \to 0$ as $t \to \infty$, because $\liminf_{t\to\infty} |\beta(t)| \neq 0$, and $\exp(-F(t))$ is monotone for $t \geqslant T$; if it diverges to $+\infty$, then again $s(t) \to 0$ as $t \to \infty$ because the numerator in (41) is bounded. This completes the proof of Theorem 1.

In the present paper, we have discussed special cases of Theorem 1(a) in Section 3 for cylindrical diverging and converging expansive waves, with $\mu = c_0 v/(2(x_0 + c_0 t))$ and $\beta = -\Lambda_0$, when $\Lambda_0 > 0$.

THEOREM 2. *Consider Eq.* (38), *with* $s(0) \neq 0$. *Suppose that* $\mu(t)$ *and* $\beta(t)$ *are integrable on every finite subinterval of* $[0, \infty)$, *the function* $\beta(t)$ *is of constant sign, and* $\operatorname{sgn} s(0) = \operatorname{sgn} \beta(t)$. *Let* $F(t) = \int_0^t \mu(\tau)\, d\tau$ *and*

$$\alpha = \left[\int_0^\infty |\beta(t)| \exp(-F(t))\, dt \right]^{-1}.$$

(i) *If* $|s(0)| > \alpha$, *there exists a unique finite time* $t_\infty > 0$ *such that*

$$\int_0^{t_\infty} \beta(t) \exp(-F(t))\, dt = 1/s(0); \qquad \textit{and} \quad \lim_{t \to t_\infty} |s(t)| = \infty.$$

(ii) *Let* $\int_0^\infty |\beta(t)|\, dt = \infty$. *If* $|s(0)| < \alpha$, *then* $\liminf_{t \to \infty} |s(t)| = 0$.

(iii) *If* $|s(0)| = \alpha$, $\liminf_{t \to \infty} |\beta(t)| \neq 0$ *and* $\mu(t)/\beta(t)$ *tends to a finite non-zero or infinite limit as* $t \to \infty$, *then* $\lim_{t \to \infty} s(t) = \lim_{t \to \infty} [\mu(t)/\beta(t)]$.

Proof. Statements (i) and (ii) form Theorem 3.2.4 in Chen; we, therefore prove (iii). The solution (39), when $\operatorname{sgn} s(0) = \operatorname{sgn} \beta(t)$, may be written as

$$s(t) = \{\operatorname{sgn} s(0)\}\{\exp(-F(t))\} \Big/ \left\{ 1/|s(0)| - \int_0^t [|\beta(\tau)| \exp(-F(\tau))]\, d\tau \right\}. \tag{42}$$

The integral in the denominator increases from 0 to $\tilde{\alpha}^1 (= |s(0)|^{-1})$ as t increases from 0 to ∞, and the integral converges over $[0, \infty)$ because $\alpha = |s(0)|$ so that $0 < \alpha < \infty$. Since μ/β tends to a finite non-zero or infinite limit as $t \to \infty$, and $\liminf_{t \to \infty} |\beta(t)| \neq 0$, it follows as in the proof of Theorem 1 that $\exp(-F(t)) \to 0$ as $t \to \infty$. Thus, both the numerator and the denominator in (42) tend to zero as $t \to \infty$, and we may apply l'Hospital's rule to obtain

$$\lim_{t \to \infty} s(t) = \{\operatorname{sgn} s(0)\} \lim_{t \to \infty} \frac{-\mu(t)}{-|\beta(t)|} = \lim_{t \to \infty} \frac{\mu(t)}{\beta(t)},$$

because $\operatorname{sgn} s(0) = \operatorname{sgn} \beta(t)$. This completes the proof.

Section 3. II(ii) of the present paper deals with a special case of Theorem 2, with $\mu = c_0 \nu / \{2(-|x_0| + c_0 t)\}$, $\beta = -\Lambda_0$, and $\alpha = s_c$.

Theorem 3.2.6 stated in Chen [6, p. 255], needs an essential modification in its statement and its proof. The statement "if there exists a finite time $t^* > 0$ such that $\lim_{t \to t^*} \mu(t) = -\infty$, then $\lim_{t \to t^*} \int_0^t \mu(\tau)\, d\tau = -\infty$" appearing in the proof is clearly incorrect because unbounded functions can be integrable over finite intervals; a standard example is the function $\mu(t) = -(1-t)^{-p}$, $0 \leqslant t < 1 = t^*$, where $0 < p < 1$. Another inaccuracy in the same proof is the last step, where it is asserted that if the integrals in the numerator and denominator of (41) are unbounded as $t \to t^*$, then so is $\mu|\beta$.

It is possible that $|\beta|$ may tend to infinity faster than μ as $t \to t^*$ so that μ/β could tend to a finite limit (e.g., consider the case $\beta = \mu^2$). The modified theorem is as follows.

THEOREM 3. *Consider the equation* (38) *with* $s(0) \neq 0$. *Suppose that* $\mu(t)$ *and* $\beta(t)$ *are integrable on every finite subinterval of* $[0, t^*)$, *the function* $\beta(t)$ *is of constant sign on* $[0, t^*)$, $\operatorname{sgn} s(0) = -\operatorname{sgn} \beta(t)$, *and*

$$\lim_{t \to t^*} \int_0^t \mu(\tau)\, d\tau = -\infty. \tag{43}$$

Let $F(t) = \int_0^t \mu(\tau)\, d\tau$ *and* $\delta = \int_0^{t^*} |\beta(t)| \exp(-F(t))\, dt$. *Then* (i) $\lim_{t \to t^*} |s(t)| = \infty$ *if* $\delta < \infty$; (ii) $\lim_{t \to t^*} |s(t)| = \lim_{t \to t^*} [\mu(t)/\beta(t)]$, *if* $\delta = \infty$ *and* $\mu(t)/\beta(t)$ *tends to a finite or infinite limit as* $t \to t^*$.

If Condition (43) *is replaced by the condition*

$$\int_0^{t^*} \mu(t)\, dt = +\infty, \qquad \text{then} \quad \lim_{t \to t^*} s(t) = 0.$$

Proof. If (43) holds, the numerator of (41) tends to ∞ as $t \to t^*$. In case (i) the denominator is bounded so that $|s(t)| \to \infty$ as $t \to t^*$. In case (ii), we may apply l'Hospital's rule as in Theorem 1.

The last statement follows because the denominator in (41) is bounded away from zero, whereas the numerator tends to zero as $t \to t^*$. This concludes the proof of the theorem.

Remark. The above theorems, when applied to our problem, yield the conditions under which the wave ultimately damps out, or takes a stable wave form, or, forms a shock or a focus. However, in the present case, the same information is obtained directly from Solution (23). Since we could extract much more information about the dependence of steepening or flattening of the wave front on s_0, γ, k_0, and ε_0, we preferred not to use the incomplete theorems of Chen [6].

The Bernoulli equation (38) appears in many situations where μ and β may not be as simple as in our problem; it is necessary to have the above modified theorems for use in more general situations.

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